Special Factors

A **perfect square** is the product of two identical factors:
$(3x^2)(3x^2) = (3x^2)^2 = 9x^4$.

A **perfect cube** is the product of three identical factors:
$(3x^2)(3x^2)(3x^2) = (3x^2)^3 = 27x^6$.

We could check any of the below formulas by multiplying them out, but notice that some of them are our Special Products from Multiplying Polynomials.

Difference of Two Squares = Product of a Sum & Difference
$a^2 – b^2 = (a + b)(a – b)$

Examples:
$4x^2 – 49y^2$ → There are only two terms, check to see if they are squares or cubes. Since they are squares with (–) in between we can factor this as the Difference of Two Squares, a = 2x, b = 7y →
$4x^2 – 49y^2 = (2x + 7)(2x – 7)$

$x^2 + 9$ → There are only two terms and they are both perfect squares, however, there is a (+) in between, we do not have a formula for factoring. If we recall our test for factorability, $b^2 – 4ac$, b = 0, a = 1, c = 9; $(0)^2 – 4(1)(9) = - 36$. Is – 36 a perfect square in Real Numbers? No, there is no Real Number such that $d^2 = - 36$, so $x^2 + 9$ is not factorable over Integers. (We will see that we can factor with Complex Numbers later, for now we cannot factor this.) So, we say that
$x^2 + 9$ is Prime.

Perfect Square Trinomial = Binomial Squared
$a^2 + 2ab + b^2 = (a + b)^2$
$a^2 - 2ab + b^2 = (a - b)^2$

Examples:
$16x^2 – 24x + 9$ → Notice the first and last terms are perfect squares, check the middle term to see if it can be written in the form ±2ab, where a = 4x & b = 3; -24x = -2(4x)(3) so this is a Perfect Square Trinomial, with a = 4x & b = 3, so →
$16x^2 – 24x + 9 = (4x – 3)^2$

$16x^2 + 24x + 9 – y^2$	The first 3 terms are the same as the previous example, so
$(4x – 3)^2 – y^2$	we know that they form a Perfect Square Trinomial.
$((4x – 3) + y)((4x – 3) – y)$	Now use the Difference of Squares from Above.
$(4x + 3 + y)(4x + 3 – y)$	Remove extra parentheses.

Sum or Difference of Two Cubes
$a^3 + b^3 = (a + b)(a^2 – ab + b^2)$
$a^3 – b^3 = (a – b)(a^2 + ab + b^2)$

Notes: A good way to remember these two formulae is "SOAP",
Same **O**pposite **A**lways **P**ositive: a^3 **S** b^3 = (a **S** b)(a^2 **O** ab **AP** b^2)
Also keep in mind, the middle term does not have a 2 in front!

Examples:
$64y^3 + 27$ → We have a y^3 & we know $27 = 3^3$, hopefully we remember that $64 = 4^3$, and so we have a sum of cubes, with a = 4y & b = 3. Using the formula we have
$64y^3 + 27 = (4y + 3)(16y^2 – 12y + 9)$.

$\frac{x^3}{8} - 125$ Yes, we have a fraction, but $8 = 2^3$, and remembering our Exponent Rules we have $\left(\frac{x}{2}\right)^3 - (5)^3$. This time we have a Difference of Cubes and can just use the above formula! Also remember when multiplying Fractions, we multiply across.

$$\left(\frac{x}{2}\right)^3 - (5)^3 = \left(\frac{x}{2} - 5\right)\left(\frac{x^2}{4} + \frac{5x}{2} + 25\right)$$

More Techniques:

Sometimes when we have 4 term polynomials we can use the Grouping Method discussed in Introduction to Factoring Polynomials, combined with a method from above.

Example:

$9x^3 + 18x^2 - 4x - 8$	First pair factor out $9x^2$, second pair factor out – 4.
$9x^2(x + 2) - 4(x + 2)$	Now factor out the $(x + 2)$
$(9x^2 - 4)(x + 2)$	Notice that $9x^2 - 4$ is a difference of squares.
$(3x + 2)(3x - 2)(x + 2)$	

Sometimes we have to use more than one of the formulas from above. Often the order of factoring matters.

Example:

$64x^6 - y^6$	First note that $2^6 = 64$.
$(2x)^6 - y^6$	Now, remember exponent rules 6 = 2*3. So we have:
$((2x)^2)^3 - (y^2)^3$	We could also write it as $((2x)^3)^2 - (y^3)^2$, we will factor later.

We will factor the difference of cubes first and see what happens.

$((2x)^2)^3 - (y^2)^3 \rightarrow (4x^2 - y^2)(16x^4 + 4x^2y^2 + y^4)$ Notice that the first factor is now a difference of squares but there is nothing we can do with the second factor. At this point try the other one to see if it factors better.

$((2x)^3)^2 - (y^3)^2 \rightarrow ((2x)^3 - y^3)((2x)^3 + y^3)$ Notice that this time we have two factors that have rules from above, Sum of Cubes and Difference of Cubes. This is the way to go! $((2x)^3 - y^3)((2x)^3 + y^3) \rightarrow (2x - y)(4x^2 + 2xy + y^2)(2x + y)(4x^2 - 2xy + y^2)$ Now we are done this problem!

When you come across a difference of powers of 6s, do the Difference of Squares then Factor the Cubes! To know which to do when takes A LOT of practice, and trial and error. If you get done factoring very early in the problem, you may want to try another way.